Teaching modes of reasoning: Redesigning the Art of Approximation in Science and Engineering

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Doubling the block’s thickness changes the note frequency by what factor?

- a. 2
- b. $\sqrt{2}$
- c. 1
- d. $1/\sqrt{2}$
- e. 1/2
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A spring model explains the doubling in frequency

Compare the stored energies for the same deflection $y$:

stored energy $\sim \text{stiffness} \times y^2$. 
A spring model explains the doubling in frequency

Compare the stored energies for the same deflection $y$: 

![Diagram showing two deflected springs with arrows indicating the deflection $y$.]
A spring model explains the doubling in frequency

Compare the stored energies for the same deflection $y$:

$4 \times$ the energy per spring
A spring model explains the doubling in frequency

Compare the stored energies for the same deflection $y$:

4× the energy per spring
2× the number of springs
A spring model explains the doubling in frequency

Compare the stored energies for the same deflection $y$:

$4 \times$ the energy per spring

$2 \times$ the number of springs

$8 \times$ the stored energy

$\text{stiffness} \times y^2$
A spring model explains the doubling in frequency

Compare the stored energies for the same deflection \( y \):

\[
\begin{align*}
4 \times & \text{ the energy per spring} \\
2 \times & \text{ the number of springs} \\
\frac{8 \times}{\text{the stored energy}} & \text{ stiffness} \times y^2 \\
\end{align*}
\]

(bending) frequency \( \sim \sqrt{\frac{\text{stiffness}}{\text{mass}}} = \sqrt{\frac{8 \times}{2 \times}} = 2 \times \).
Modes of reasoning or topics?
Modes of reasoning are a better organization than topics

Using modes of reasoning makes the course finite
Using modes of reasoning promotes transfer
Using modes of reasoning promotes long-lasting learning
Topics are many, life is short

sound
waves
mechanical properties
thermal properties
weather
fluid drag
turbulence

Where do you stop?

gravitation
prime numbers
retinal rod
biomechanics
astrophysics
financial math
…
Teaching in Cambridge, England, I moved toward modes of reasoning only subconsciously.
I was influenced by problem solving in mathematics

The Invariance Principle
Coloring Proofs
The Extremal Principle
The Box Principle
Enumerative Combinatorics
Number Theory
Inequalities
The Induction Principle
Sequences
Polynomials
Functional Equations
Geometry
Games
Further Strategies
I used it for *Street-Fighting Mathematics*

- Dimensions
- Easy cases
- Lumping
- Pictorial proofs
- Taking out the big part
- Analogy
Modes of reasoning now seemed to appear everywhere

Pt. I. Incentives
   Ex ante and ex post
   The idea of efficiency
   Thinking at the margin
   The single owner
   The least cost avoider
   Administrative cost
   Rents
   The Coase theorem

...
Modes of reasoning for science and engineering organized themselves slowly

Easy cases
Divide and conquer
Spring models
Lumping
Proportional reasoning
Symmetry/conservation
Abstraction
Probabilistic reasoning
Dimensional analysis
Modes of reasoning for science and engineering organized themselves slowly

<table>
<thead>
<tr>
<th>Abstraction</th>
<th>Easy cases</th>
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Modes of reasoning for science and engineering organized themselves slowly

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Modes of reasoning for science and engineering organized themselves around mastering complexity.
Using modes of reasoning makes the course finite
Modes of reasoning are a better organization than topics

Using modes of reasoning makes the course finite

Using modes of reasoning promotes transfer

Using modes of reasoning promotes long-lasting learning
The tree gives each mode of reasoning a place

to master complexity

organize it          discard it

divide/conquer  abstraction  lossless

lossy

symmetry/conservation  proportional reasoning  dimensional analysis

lumping  probability  easy cases  springs
Each mode of reasoning contains examples: Divide and conquer

How many barrels of oil does the United States import in a year?

\[
N_{\text{cars}} = 3 \times 10^8 \\
\text{miles/year} = 20,000 \\
\text{miles/gallon} = 30 \\
\text{gallons/barrel} = 60 \\
\text{height} = 1 \text{ m} \\
\text{width} = 0.5 \text{ m} \\
\text{depth} = 0.5 \text{ m} \\
\text{gallons/meter}^3 = 250
\]
Each mode of reasoning contains examples: Divide and conquer

How much energy does a 9-volt battery contain?

energy in a 9V battery

laptop-battery energy

laptop power

volume adjustment

battery life
Each mode of reasoning contains examples:
Divide and conquer

How much energy does a 9-volt battery contain?

- energy in a 9V battery: $2 \times 10^4$ J
- laptop-battery energy: $2 \times 10^5$ J
- volume adjustment: $1/10$
- laptop power: 20 W
- battery life: 4 hr ($10^4$ s)
Using modes of reasoning promotes transfer

mode example
Using modes of reasoning promotes transfer

When teaching by topics, it is too easy to use too-similar examples.
Using modes of reasoning promotes transfer

Diverse examples help clarify the core idea.

Using modes of reasoning automatically produces diverse examples.
Modes of reasoning are a better organization than topics

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to master complexity

organize it               discard it

divide/conquer     abstraction     lossless

symmetry/conservation proportional reasoning dimensional analysis

lossy

lumping   probability easy cases springs
How fast will the cone fall? (by conservation of energy)

$$\text{drag force} = \frac{\text{energy consumed by drag}}{\text{distance traveled}}.$$
How fast will the cone fall?
Each mode of reasoning contains examples: Symmetry and conservation

How fast will the cone fall?

\[ F \sim \rho A v^2 \]

\[ v \sim \sqrt{\frac{F}{\rho A}} \sim \sqrt{\frac{10^{-3} \text{ kg} \times 10 \text{ m/s}^2}{1 \text{ kg/m}^3 \times 0.01 \text{ m}^2}} \sim 1 \text{ m/s}. \]
The tree gives each mode of reasoning a place to master complexity

- **organize it**
  - divide/conquer
  - abstraction

- **lossless**
  - symmetry/
    - proportional
  - dimensional
    - analysis

- **lossy**
  - lumping
  - probability
  - easy cases
  - springs

- **discard it**

**abstraction**

**divide/conquer**
Each mode of reasoning contains examples:
Proportional reasoning

*Wood blocks*

![Diagram showing two wood blocks with taps and an equation: frequency ∝ thickness?](image)

*frequency ∝ thickness?*
Each mode of reasoning contains examples: Proportional reasoning

_Falling cones again_

\[
\frac{v_{\text{four stacked cones}}}{v_{\text{one stacked cone}}} = \begin{cases} 
4 \\
2 \\
\text{or} \\
\sqrt{2}
\end{cases}
\]

Equivalently,

\( v \propto (\text{number of cones}) \)?
The tree gives each mode of reasoning a place.
Each mode of reasoning contains examples:
Dimensional analysis

Motto: *The uncompared quantity is not worth knowing.*

<table>
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<td>cost of line (mains) power</td>
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Each mode of reasoning contains examples: Dimensional analysis

Motto: *The uncompared quantity is not worth knowing.*

\[
\frac{\text{cost of 9-volt battery energy}}{\text{cost of line (mains) power}} \approx \frac{\$1 \times 2 \times 10^4 \text{ J}}{\$0.15 \times 3.6 \times 10^6 \text{ J}} \\
\approx 7 \times 180 \\
\approx 1000.
\]
Each mode of reasoning contains examples:
Lumping

Every number is of the form:
\[
\begin{pmatrix}
\text{one} \\
\text{or} \\
\text{few}
\end{pmatrix} \times 10^n,
\]

where
\[\text{few}^2 = 10.\]
Each mode of reasoning contains examples: Lumping

How many seconds in a year?

\[
\frac{365 \text{ days}}{\text{year}} \times \frac{24 \text{ hours}}{\text{day}} \times \frac{3600 \text{ seconds}}{\text{hour}}
\]
Each mode of reasoning contains examples: Lumping

How many seconds in a year?

\[ \frac{\text{few} \times 10^2 \text{ days}}{\text{year}} \times \frac{24 \text{ hours}}{\text{day}} \times \frac{3600 \text{ seconds}}{\text{hour}} \]
Each mode of reasoning contains examples: Lumping

How many seconds in a year?

\[
\frac{\text{few} \times 10^2 \text{ days}}{\text{year}} \times \frac{\text{few} \times 10^1 \text{ hours}}{\text{day}} \times \frac{3600 \text{ seconds}}{\text{hour}}
\]
Each mode of reasoning contains examples: Lumping

How many seconds in a year?

\[
\frac{\text{few} \times 10^2 \text{ days}}{\text{year}} \times \frac{\text{few} \times 10^1 \text{ hours}}{\text{day}} \times \frac{\text{few} \times 10^3 \text{ seconds}}{\text{hour}}
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Each mode of reasoning contains examples: Lumping

How many seconds in a year?

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\frac{\text{few} \times 10^2 \text{ days}}{\text{year}} \times \frac{\text{few} \times 10^1 \text{ hours}}{\text{day}} \times \frac{\text{few} \times 10^3 \text{ seconds}}{\text{hour}}
\]

\[
\approx \text{few} \times 10^7 \frac{\text{seconds}}{\text{year}}
\]
The tree gives each mode of reasoning a place

to master complexity

organize it  discard it

divide/conquer  abstraction  lossless

lossy

symmetry/conservation  proportional reasoning  dimensional analysis

lumping  probability  easy cases  springs
Each mode of reasoning contains examples:
Springs

Wood blocks
Each mode of reasoning contains examples:

Springs

Why is the sky blue?

How much energy does the earth–sun system lose in gravitational radiation?
Connections are more important than facts alone

big cluster $= 12\%$

$p_{bond} = 0.40$
Connections are more important than facts alone

- Big cluster = 12%
  \[ p_{\text{bond}} = 0.40 \]

- Big cluster = 42%
  \[ p_{\text{bond}} = 0.50 \]
Connections are more important than facts alone

big cluster = 12%
$p_{\text{bond}} = 0.40$

big cluster = 42%
$p_{\text{bond}} = 0.50$

big cluster = 81%
$p_{\text{bond}} = 0.55$
Connections are more important than facts alone

- Big cluster = 12\% 
  - $p_{\text{bond}} = 0.40$

- Big cluster = 42\% 
  - $p_{\text{bond}} = 0.50$

- Big cluster = 81\% 
  - $p_{\text{bond}} = 0.55$

- Big cluster = 94\% 
  - $p_{\text{bond}} = 0.60$
Using modes of reasoning promotes long-lasting learning
Modes of reasoning are a better organization than topics

Using modes of reasoning makes the course finite

Using modes of reasoning promotes transfer

Using modes of reasoning promotes long-lasting learning
The goal [of teaching] should be, not to implant in the students’ mind every fact that the teacher knows now; but rather to implant a way of thinking that enables the student, in the future, to learn in one year what the teacher learned in two years. Only in that way can we continue to advance from one generation to the next.

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Produced with free software:
PDFTEX, ConTeXt, Python, and MetaPost

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